Numerical Approximation of an Optimum Growth Program

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Introduction

Economics of growth deals with optimal intertemporal allocation of scarce resources. To consider it a problem of dynamic optimization one has to realize the dual character of consumption in an intertemporal framework. If present output can either be consumed or saved, which implies a transfer to future resources, a more of consumption today will lessen future output and hence future consumption possibilities. Less consumption today instead will imply forgone instantaneous utility for the sake of investment. Hence solving for the optimum amount of present consumption is a key issue of economic modeling.

The problem was first treated by Ramsey (1928). Koopmans (1965) and especially Cass (1965) explored Ramsey's classic approach by applying the *maximum principle*, which is a powerful dynamic optimization tool for economic problems (Shell, 1969). This work lead to the now well-known textbook model of optimal growth (Acemoglu, 2009; Chiang, 1992).

Since the existence of an optimum consumption trajectory is shown, we actually want to approximate a numerical solution. For this purpose we use a

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technique named *eigendecomposition of a matrix* (Novales, Fernández, and Ruiz, 2009). The solution algorithm enables us to simulate optimum growth trajectories for different initial conditions or exogenous shocks.

Model

Consider the dynamical system described by (1)→(4)

$$\max_{c}\ \mathcal{W}(k(0)) = \int_0^{\infty} u(c)e^{-rt}\mathrm{d}t \tag{1}$$

$$\text{s.t.}\quad \frac{\mathrm{d}k(t)}{\mathrm{d}t} =: \dot{k} = f(k) - c - (n+\delta)k \tag{2}$$

$$k_0 = k(0) \tag{3}$$

$$c \in [0, f(k)] \tag{4}$$

where t is the time index. The objective functional (welfare integral) is denoted by $\mathcal{W}(\cdot)$. It is maximized over the control variable c, which represents per worker consumption. The society's utility function $u(\cdot)$ is twice continuous differentiable with respect to c and strictly concave, i.e. $u'(c) > 0,\ u''(c) < 0$. Future consumption utilization is discounted by the time preference rate $r > 0$. The law of motion for capital accumulation is given by (2), where k is the capital labor ratio. The linear homogeneous production function $f(\cdot)$, which represents the output per worker $f(k) = y$, is twice continuous differentiable with respect to k and strictly concave, i.e. $f'(k) > 0,\ f''(k) < 0$. The growth rate of the labor force[1] is denoted by n and $\delta \in [0, 1]$ represents the depreciation rate of capital. The initial condition is given by $k(0) = k_0$. The control variable is restricted by (4). The problem of the social planer is to maximize society's utility from present and future consumption flows by controlling the state k over c.

[1]For simplicity we assume that labor force equals population.

Dynamic Optimization The dynamic optimization problem given in (1) can be solved by the maximum principle of Pontrjagin, Boltjanski, Gamqrelidse, and Mishchenko (1964). This method refers to the Hamiltonian approach to dynamic economics (Cass and Shell, 1976).

Definition 1 (Hamiltonian). $\mathcal{H}(t, k, c, \mu) := e^{-rt}\{u(c) + \mu[f(k) - c - (n + \delta)k]\}$

where μ is the current value costate variable, which expresses the shadow price for an extra unit of capital at time t.[2] The shadow price therefore "translates" a change of the capital stock in a change of utility. Thus the Hamilton function measures total utility flows, which are derived directly by instantaneous utility $u(\cdot)$ and indirectly by the change of capital $\mu\dot{k}$. The maximum principle provides the following first order conditions[3]

$$\mathcal{H}_c = u'(c) - \mu = 0, \tag{5}$$

$$\dot{\mu} = -\mathcal{H}_k e^{rt} + r\mu = -\mu[f'(k) - (n + \delta + r)]. \tag{6}$$

where $\mathcal{H}_c$ is the partial derivative of the Hamiltonian with respect to c and $\mathcal{H}_k$ with respect to k respectively.[4] For an economic reasoning of these conditions see Dorfman (1969). Differentiating (5) with respect to t yields

$$\dot{\mu} = u''(c)\dot{c}. \tag{7}$$

[2] While the present value multiplier $\lambda e^{rt} := \mu$ measures the shadow price of an extra unit of capital at time $t = 0$.

[3] The conditions of the maximum principle are necessary and sufficient for a maximization, if the maximized Hamiltonian, defined as $\mathcal{H}^0(t, k, \mu) := \max_c \mathcal{H}(t, k, c, \mu)$, is concave in k as stated by Arrow (1968) and formally proved by Seierstad and Sydsæter (1977).

[4] For completion we have to impose two transversality conditions $\lim_{t\to\infty} \mu(t)e^{-rt} = \lim_{t\to\infty} \mathcal{H}(t) = 0$.

Combining and rearranging (5), (6) and (7) finally gives the *Euler equation*

$$\dot{c} = \frac{c}{\sigma}[f'(k) - (n + \delta + r)] \tag{8}$$

where $\sigma := -cu''(c)/u'(c) > 0$[5] is the measure of intertemporal risk aversion.

System Dynamics and Phase Diagram The differential equations (2) and (8) are known to be a two dimensional system of differential equations which describe the dynamics of the state and control variable in the (k, c)-plane. The possible phase transitions are described by the following cases

$$\dot{k} \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} 0 \quad \text{if} \qquad c \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} f(k) - (n + \delta)k \tag{9}$$

$$\dot{c} \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} 0 \quad \text{if} \quad f'(k) \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} n + \delta + r \tag{10}$$

Figure 1(a) refers to the general system dynamics. We call the two loci $\dot{k} = 0$ and $\dot{c} = 0$ phase boundaries, which seperate the $\mathbb{R}^2_+ := \{(k, c)|k, c \geq 0\}$ in single phase regions where phase transitions occur. The arrows indicate the transition direction of capital and consumption over time. A dynamic equilibrium is located at the intersection of $\dot{k} = \dot{c} = 0$, which is formally defined as a fixed point.

Definition 2. *A fixed point* $E(\tilde{k}, \tilde{c})$ *is an equilibrium point in the* (k, c)*-plane, such that* $\lim_{t\to\infty} \dot{k}(t) = \dot{c}(t) = 0$ *holds.*

Figure 1(b) shows phase lines for different initial choices of $c(0)$, where $k(0)$ is given exogenously. There exists one pair of phase lines which flow towards the

[5]The strict concavity of $u(\cdot)$ implies $\lim_{c\downarrow 0} u'(c) \to \infty$, i.e. $c > 0$.

fixed point and is indicated by the broken lines $(- \cdot -)$. The "stable branch" is known to be the *balanced growth path*. With a given $k(0)$ there exists only one optimal choice of $c^*(0)$ which leads on the optimal growth path. A different choice of $c(0)$ will lead to an explosive growth of either c or k while the other variable shrinks to zero.[6]

Computation of an Optimum Growth Path

The fixed point property of $\dot{c} = 0$ yields $f'(\tilde{k}) = n + \delta + r$. Consider a Cobb-Douglas production function $f(k) = k^\alpha$ with $\alpha \in (0, 1)$ representing the prodcution elasticity of capital. The steady state value for capital is then given by

$$\begin{aligned} \alpha\tilde{k}^{\alpha-1} &= n + \delta + r \\ \Rightarrow \tilde{k} &= \left(\frac{\alpha}{n+\delta+r}\right)^{\frac{1}{1-\alpha}}. \end{aligned} \tag{11}$$

In addition a fixed point implies $\dot{k} = 0$, that is

$$\tilde{c} = \tilde{k}^\alpha - (\delta + n)\tilde{k}. \tag{12}$$

The $\ln(\cdot)$ of (2) and (8) are

$$\frac{\mathrm{d}\ln k}{\mathrm{d}t} = e^{(\alpha-1)\ln k} - e^{\ln c - \ln k} - (n + \delta) =: \Pi(k, c) \tag{13}$$

$$\frac{\mathrm{d}\ln c}{\mathrm{d}t} = \frac{1}{\sigma}\left[\alpha e^{(\alpha-1)\ln k} - (n + \delta + r)\right] =: \Psi(k, c) \tag{14}$$

[6]Formally one cannot neglect the complementary slackness condition eq. (4), i.e. consumption is only feasible if there is a positive capital stock. Hence the critical value $c = f(k)$ leads the growth path instantly to the repellor point $(0, 0)$.

A functional relation between consumption and capital is given by (see technical appendix p. 97)

$$\ln c(t) = \frac{\xi}{\omega_2}(\ln k(t) - \ln \tilde{k}) + \ln \tilde{c}, \quad \forall t \in [0, \infty). \tag{15}$$

where $\xi := \Psi(\tilde{k}, \tilde{c})_{\ln k} = \dfrac{(\alpha - 1)(n + \delta + r)}{\sigma}$ is an auxiliary variable and $\omega_2 = 0.5 \cdot \left(r - \sqrt{r^2 - 4\psi\xi}\right)$ is the second characteristic root with the second auxiliary variable $\psi := -\Pi(\tilde{k}, \tilde{c})_{\ln c} = \dfrac{(1 - \alpha)(n + \delta) + r}{\alpha}$. Notice that consumption is now determinated by capital and a given $k(0) = k^*(0)$ thus yields $c^*(0)$. By the law of motion for capital (2) one might iterate the optimum trajectories for the model variables.

Simulation

In the spirit of Kendrick and Taylor (1971) and Islam (2001) we now compute optimal trajectories for model economies. The structural parameters of the benchmark economy are calibrated as follows $\alpha = 0.3$, $\delta = 0.2$, $\sigma = 2.0$, $r = 0.04$ and $n = 0.0$.[7]

Different Initial Conditions Figure 2(a) shows that two different initial capital stocks $k_1(0) > k_2(0)$ are given. First we analyse the general pattern, which both trajectories refer to. Assume the given capital stock $k(0)$ is lower than the long term optimal stock $\lim_{t\to\infty} k^*(t) = \tilde{k}$, which is derivable by the production function and the given structural parameters (see eq. (11)). The corresponding per capita consumption $\tilde{c}$ is obtained by (12). To force the economy on the stable growth path, which tends towards the dynamic equilibrium $(\tilde{k}, \tilde{c})$, we derive optimal initial consumption $c^*(0)$ by (15). Figure 2 shows the transition dynamics for

[7] The matlab source code for the considered simulations is available at `http://bje.uni-bonn.de`

capital, consumption, production, investments, instantaneous utility and current value shadow price. Initial consumption is lower than production $c(0) < y(0)$, i.e. gross investment is positive $i(0) > 0$. If gross investment outweighs depreciated capital, we have positive net investment, i.e. $i(0) - \tilde{i} > 0$ and the economy is accumulating capital. Hence the production in the next period will increase. The forgoing consumption today will increase future consumption potential and utility as well. Since the utility function is concave, the marginal utility of an additional consumption unit decreases as well as the shadow price of capital. That is the social planer is willing to give up consumption for an extra unit of capital if the consumption level is relatively high. The process of capital accumulation continues until the economy reaches the fixed point, where the social planer has no incentive to increase capital, because the future utility gain does not compensate present consumption abstinence. To keep production constant the planer uses a fraction of output to cover just the depriciated capital, i.e. the economy is in steady state. In addition it can be shown that the two transversality conditions $\lim_{t\to\infty} \mu(t)e^{-rt} = \mathcal{H}(t) = 0$ hold, as stated by the maximum principle.

Figure 2 shows a counterintuitively result. One may wonder that an economy with a relatively large initial capital stock will converge to steady state more rapidly than an economy with a relatively lower one. However figures 2(a) $\rightarrow$ 2(f) show that those trajectories adjust to each other over time and reach steady state at the same time. This phenomenon is describable as follows: a given capital stock instantly determines initial consumption by (15). Production is given by $f(k(0)) = y(0)$ and dued to $i(0) = y(0) - c(0)$ initial investment is given as well. Figure 2(d) shows that investment of economy *2* excess economy *1*, i.e. $i_1(0) < i_2(0)$. Furthermore the extra amount of capital will be used more efficiently, since marginal productivity is relatively higher on a lower capital stock $f'(k_1(1)) < f'(k_2(1))$. As shown in figure 2(c) we might figure that by the steeper slopes of $\dot{y}_1 < \dot{y}_2$. Consequently consumption and corresponding utility increase faster (cf. fig. 2(b) and 2(e)). At the end the investment dynamic results in a successive adjustment over time until the fixed point is reached.

Impulse Response Consider a steady state economy in which all macroeconomic variables are quasi-constant. We now simulate a shock by fixing all structural parameters but increase the time preference ($r \uparrow$) at $t = 10$. The planer now values present consumption more than future consumption. Figure 3(b) shows that short-run consumption increases as well as instantaneous utility (cf. 3(e)). The more of consumption is possible at the expense of investment, which explains the slump in figure 3(d). The heavy decline of investment in the short-run is known as overshooting, since the reaction is stronger than long term development. It follows that future capital stock and simultaneously production shrink (cf. fig. 3(a) and 3(c)). On the other hand a higher consumption level leads to a lower shadow price for capital, since it is measured in marginal utility (cf. fig. 3(f)). This effect balances the shrinking process such that investment recovers and production is stabilized. In the long run the steady state values for capital, consumption, output, investment and utility are lower than before the shock. That is just the result of valuing present consumption at the expense of capital accumulation and future consumption potential.

Conclusion

The present paper shows an easy-to-use solving technique for dynamic optimization problems. The solution algorithm enables the user to approximate optimal trajectories for the state, costate and control variable. One might adjust and/or add structural parameters to solve various problems.

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Appendix

Figures

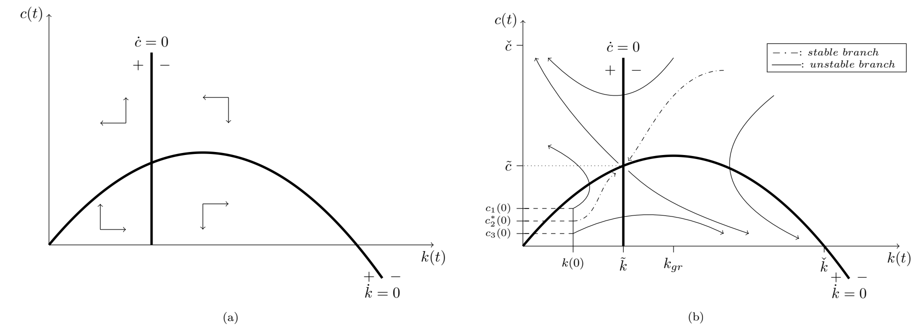
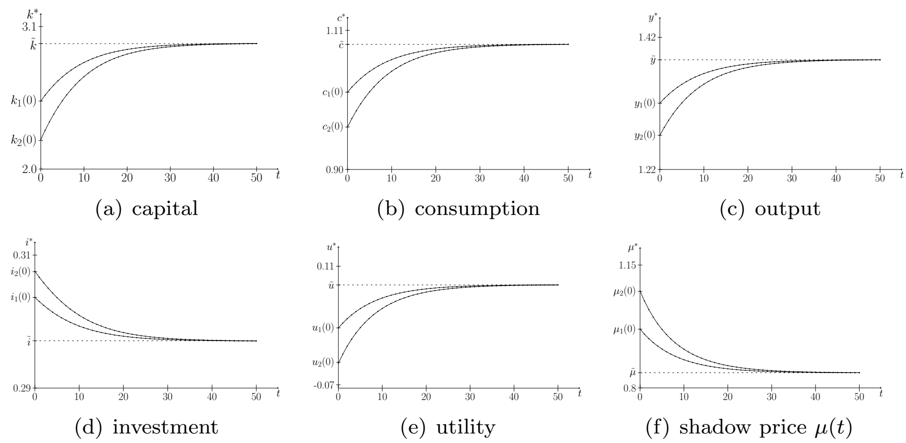


Figure 1: Phase Diagram for Consumption and Capital Dynamics

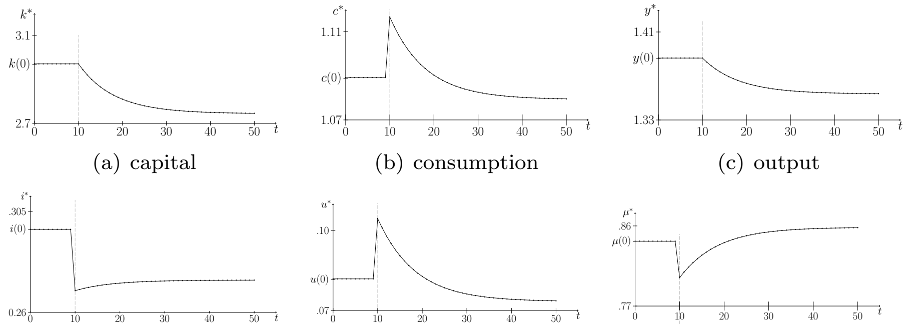


(a) capital (b) consumption (c) output

(d) investment (e) utility (f) shadow price $\mu(t)$

Note: These figures show the optimal trajectories of the considered macroeconomic variables for $t \in [0, 50]$ *and for two different initial conditions* $k_i(0) < \tilde{k}$, $i = 1, 2$. *From* $t > 50$ *both model economies have adjusted to the fixed point, i.e. all variables are constant, unless one or more structural parameters are varied.*

Figure 2: Transition to Fixed Point for different $k(0) < \tilde{k}$



(a) capital (b) consumption (c) output

(d) investment (e) utility (f) shadow price $\mu(t)$

Note: These figures show the optimal impulse response of the considered macroeconomic variables due to an permanent increase of the time preference rate. The economy is adjusting to a new dynamic equilibrium when the shock is imposed at $t = 10$.

Figure 3: Optimal Impulse Response on Permanent Change in r

Eigendecomposition of a Matrix

We can establish following fixed point properties by setting (14) and (13) equal to zero

$$
\begin{aligned}
e^{(\alpha-1)\ln\tilde{k}} &= \frac{n+\delta+r}{\alpha} \\
e^{\ln\tilde{c}-\ln\tilde{k}} &= e^{(\alpha-1)\ln\tilde{k}} - (n+\delta) = \frac{n+\delta+r}{\alpha} - (n+\delta) \\
&= \frac{(1-\alpha)(n+\delta)+r}{\alpha} := \psi > 0
\end{aligned}
$$

(13) and (14) can be considered as a two dimensional differential equation system, where its linearization around the fixed point is provided by a first order Taylor series, i.e.

$$
\begin{bmatrix} \Pi \\ \Psi \end{bmatrix}_E \cong \begin{bmatrix} \Pi_{\ln k} & \Pi_{\ln c} \\ \Psi_{\ln k} & \Psi_{\ln c} \end{bmatrix}_{(\tilde{k},\tilde{c})} \begin{bmatrix} \ln k(t) - \ln\tilde{k} \\ \ln c(t) - \ln\tilde{c} \end{bmatrix}
$$

The partial derivatives at the fixed point are

$$
\begin{aligned}
\frac{\partial \Pi(\tilde{k},\tilde{c})}{\partial \ln k} &= (\alpha-1)e^{(\alpha-1)\ln \tilde{k}} + e^{\ln \tilde{c} - \ln \tilde{k}} \\
&= \frac{(\alpha-1)(n+\delta+r) + (1-\alpha)(n+\delta) + r}{\alpha} = r \\
\frac{\partial \Pi(\tilde{k},\tilde{c})}{\partial \ln c} &= -e^{\ln \tilde{c} - \ln \tilde{k}} = -\psi < 0 \\
\frac{\partial \Psi(\tilde{k},\tilde{c})}{\partial \ln k} &= \frac{\alpha(\alpha-1)}{\sigma} e^{(\alpha-1)\ln \tilde{k}} = \frac{(\alpha-1)(n+\delta+r)}{\sigma} =: \xi < 0 \\
\frac{\partial \Psi(\tilde{k},\tilde{c})}{\partial \ln c} &= 0
\end{aligned}
$$

Such that finally results

$$
\underbrace{\begin{bmatrix} \Pi \\ \Psi \end{bmatrix}}_{\dot{h(t)}} \cong \underbrace{\begin{bmatrix} r & -\psi \\ \xi & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \ln k(t) - \ln \tilde{k} \\ \ln c(t) - \ln \tilde{c} \end{bmatrix}}_{h(t)} \tag{16}
$$

With eigenvalues ω_1 und ω_2

$$
\omega_1,\ \omega_2 = \frac{r \pm \sqrt{r^2 - 4\psi\xi}}{2}.
$$

Since $\omega_1\omega_2 = \det(A) = \psi\xi < 0$ the equilibrium point is a saddlepoint. For later reasoning we note $\omega_1 > r > 0$ and $\omega_2 < 0$. Let $\mathbf{j}' = (j_1, j_2)'$ the eigenvector of ω_1 and $\mathbf{m}' = (m_1, m_2)'$ of ω_2 respectively, i.e.

$$
A \begin{bmatrix} j_1 \\ j_2 \end{bmatrix} = \omega_1 \begin{bmatrix} j_1 \\ j_2 \end{bmatrix} \qquad \text{and} \qquad A \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \omega_2 \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}.
$$

Normalizing $j_1 = m_1 = 1$ yields for j_2 and m_2

$$j_2 = \frac{\omega_1 - r}{-\psi} = \frac{\xi}{\omega_1} \qquad \text{and} \qquad m_2 = \frac{\omega_2 - r}{-\psi} = \frac{\xi}{\omega_2}.$$

Let G be a matrix which contains as its columns the eigenvecotrs of A, G^{-1} its inverse and D a diagonal matrix, which elements are the eigenvalues of A, i.e.

$$G = \begin{bmatrix} j_1 & m_1 \\ j_2 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{\xi}{\omega_1} & \frac{\xi}{\omega_2} \end{bmatrix}$$

$$G^{-1} = \begin{bmatrix} j_1 & m_1 \\ j_2 & m_2 \end{bmatrix}^{-1} = \frac{1}{j_1 m_2 - j_2 m_1} \begin{bmatrix} m_2 & -m_1 \\ -j_2 & j_1 \end{bmatrix} = \frac{\omega_1 \omega_2}{\xi(\omega_1 - \omega_2)} \begin{bmatrix} \frac{\xi}{\omega_2} & -1 \\ -\frac{\xi}{\omega_1} & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix}$$

We now factorize A to $A = GDG^{-1}$

$$A = \frac{\omega_1 \omega_2}{\xi(\omega_1 - \omega_2)} \begin{bmatrix} 1 & 1 \\ \frac{\xi}{\omega_1} & \frac{\xi}{\omega_2} \end{bmatrix} \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix} \begin{bmatrix} \frac{\xi}{\omega_2} & -1 \\ -\frac{\xi}{\omega_1} & 1 \end{bmatrix}$$

The compact notation of (16) is

$$\dot{h(t)} \cong Ah(t). \tag{17}$$

Since D is the Jordan canonical form of A, (17) can be solved with regard to the matrix exponential and a given initial condition $h(0)$

$$h(t) \cong e^{At} h(0) = e^{GDG^{-1}t} h(0) = Ge^{Dt} G^{-1} h(0)$$

that is

$$\begin{bmatrix} \ln k(t) - \ln \tilde{k} \\ \ln c(t) - \ln \tilde{c} \end{bmatrix} \cong \frac{\omega_1 \omega_2}{\xi(\omega_1 - \omega_2)} \begin{bmatrix} 1 & 1 \\ \frac{\xi}{\omega_1} & \frac{\xi}{\omega_2} \end{bmatrix} \begin{bmatrix} e^{\omega_1 t} & 0 \\ 0 & e^{\omega_2 t} \end{bmatrix} \begin{bmatrix} \frac{\xi}{\omega_2} & -1 \\ -\frac{\xi}{\omega_1} & 1 \end{bmatrix} \begin{bmatrix} \ln k(0) - \ln \tilde{k} \\ \ln c(0) - \ln \tilde{c} \end{bmatrix}$$

alternatively written as a system of equations

$$\begin{aligned} \ln k(t) - \ln \tilde{k} &= e^{\omega_1 t} b_{11} + e^{\omega_2 t} b_{12} \\ \ln c(t) - \ln \tilde{c} &= e^{\omega_1 t} b_{21} + e^{\omega_2 t} b_{22} \end{aligned} \tag{18}$$

with

$$\begin{aligned} b_{11} &= \quad \frac{\omega_1}{\xi(\omega_1 - \omega_2)} [\xi(\ln k(0) - \ln \tilde{k}) - \omega_2(\ln c(0) - \ln \tilde{c})], \\ b_{12} &= -\frac{\omega_2}{\xi(\omega_1 - \omega_2)} [\xi(\ln k(0) - \ln \tilde{k}) - \omega_1(\ln c(0) - \ln \tilde{c})], \\ b_{21} &= \quad \frac{1}{\omega_1 - \omega_2} [\xi(\ln k(0) - \ln \tilde{k}) - \omega_2(\ln c(0) - \ln \tilde{c})], \\ b_{22} &= \quad -\frac{1}{\omega_1 - \omega_2} [\xi(\ln k(0) - \ln \tilde{k}) - \omega_1(\ln c(0) - \ln \tilde{c})]. \end{aligned}$$

The left hand side of (18) provides the deviation of the current value from the steady state. For an optimal choice of $c(0)$ the systems tends for $t \to \infty$ on the stable growth path towards its fixed point. By definition the deviation equals zero at the fixed point and hence stability requires

$$\lim_{t \to \infty} = \ln k(t) - \ln \tilde{k} = \ln c(t) - \ln \tilde{c} = 0$$

With $\omega_2 < 0$ we get

$$\lim_{t \to \infty} = e^{\omega_2 t} b_{12} = e^{\omega_2 t} b_{22} = 0.$$

And since $\omega_1 > 0$, we set $b_{11} = b_{21} = 0$

$$\begin{aligned}\xi(\ln k(0) - \ln \tilde{k}) &= \omega_2(\ln c(0) - \ln \tilde{c})\\ \Rightarrow \ln c(0) &= \frac{\xi}{\omega_2}(\ln k(0) - \ln \tilde{k}) + \ln \tilde{c}. \end{aligned} \tag{19}$$

This equation determines the optimum initial consumption $c^*(0)$ for a given stock of capital. Substitute (19) in b_{12} and b_{22} yields

$$\begin{aligned} b_{12} &= -\frac{\omega_2}{\xi(\omega_1 - \omega_2)}\left\{\xi(\ln k(0) - \ln \tilde{k}) - \omega_1\left[\frac{\xi(\ln k(0) - \ln \tilde{k})}{\omega_2}\right]\right\}\\ &= \ln k(0) - \ln \tilde{k},\\ b_{22} &= -\frac{1}{\omega_1 - \omega_2}\left\{\xi(\ln k(0) - \ln \tilde{k}) - \omega_1\left[\frac{\xi(\ln k(0) - \ln \tilde{k})}{\omega_2}\right]\right\}\\ &= \frac{\xi}{\omega_2}(\ln k(0) - \ln \tilde{k}). \end{aligned}$$

The solution of (18) for a determined choice of $c(0)$ by $k(0)$ is given by

$$\begin{aligned}\ln k(t) - \ln \tilde{k} &= e^{\omega_2 t}(\ln k(0) - \ln \tilde{k})\\ \ln c(t) - \ln \tilde{c} &= e^{\omega_2 t}\frac{\xi}{\omega_2}(\ln k(0) - \ln \tilde{k})\end{aligned}$$

The stability condition for the initial consumption (19) is presumed to be binding, then for every instant of time the relation between consumption and capital equals

$$\ln c(t) = \frac{\xi}{\omega_2}(\ln k(t) - \ln \tilde{k}) + \ln \tilde{c}, \quad \forall t \in [0, \infty).$$