Econometric Tests for Speculative Bubbles

Prof. Dr. Jörg Breitung [*]

Introduction

Speculative bubbles have a long history in stock, commodity or real estate markets. Famous historical examples include the Dutch Tulipmania (1634-1637) and the Mississippi Bubble (1719–1720). Such bubbles may lead to severe economic crises such as the speculative excess on stock prices prior to the Great Depression from 1930–1933 or the recent financial crisis of 2007–2009 that was preceded by the US housing bubble.There is a broad consensus that bubbles are characterized by an explosive path of the underlying market prices, whereas in "normal times", speculative prices are well approximated by a random walk process. A standard economic model to motivate the occurrence and persistence of speculative bubbles is the framework of rational bubbles (see, e.g., Blanchard and Watson (1982) and Flood and Hodrick (1990)). In such models it is economically rational to invest in an obviously overpriced asset as long as the investor asserts that the price continues to rise exponentially.

The theory of rational bubbles commonly starts from a simple present value model for asset prices (see, e.g. Campbell, Lo, and MacKinlay (1997)). If in-

[*]Faculty of Management, Economics and Social Sciences , Institute for Econometrics and Statistics, University of Cologne, Meister-Ekkehart-Str. 9 Building: 113, D-50923 Cologne, Germany, e-mail: breitung@statistik.uni-koeln.de, phone: +49-221-470-4266

vestors are not averse to risk, all assets would have the same constant expected

real return R in equilibrium that is

$$1 + R = E_t \left(\frac{P_{t+1} + D_{t+1}}{P_t} \right), \tag{1}$$

where $E_t(\cdot)$ denotes the expected value given the information in period t. This

difference equation can be solved by forward iteration yielding

$$P_t = P_t^f + B_t, \tag{2}$$

$$\text{with } P_t^f = E_t(P_{t+1}^*) \equiv \sum_{i=1}^{\infty} \left(\frac{1}{1+R} \right)^i E_t(D_{t+i}). \tag{3}$$

$$\text{and } E_t(B_{t+1}) = (1+R)B_t. \tag{4}$$

The bubble component B_t arises from the fact that the solution of a difference

equation is not unique. The component P_t^f is often referred to as the fundamen-

tal (stock) price which equals the expected present value of the future dividend

payments, whereas equation (4) is a no arbitrage condition for the so-called bub-

ble component of the stock price. If a bubble is present in the stock price, (4)

requires that a rational investor, who is willing to buy that stock, must expect

the bubble to grow at a rate equal to R. Excluding negative stock prices one can

infer that $B_t \geq 0$ for all t. Whenever $B_t > 0$, a rational investor is willing to buy

an "overpriced" stock, since (s)he believes that through price increases (s)he will

be sufficiently compensated for the extra payment due to the bubble component

(B_t). If the bubble component is a large part of the price, then the expectation

that it will increase at rate R means that investors expect price increases that

have nothing to do with changes in fundamentals. If enough investors have this

expectation and buy shares, the stock price will indeed go up and complete a loop of a self-fulfilling prophecy. Obviously, it does not make sense to assume a bubble to continue infinitely long, since in this case the price of the asset will tend to infinity at an exponential rate. Brock (1982) analyzed the maximization problem of a competitive, representative, infinitely-lived investor and obtained a terminal condition (known as a transversality condition) that allows to exclude rational bubbles. Another theoretical challenge against rational bubbles is provided by Diba and Grossman (1988) who note that bubbles in real stock prices can never be negative. Since the fundamental value with nonzero dividend payment must grow with a lower rate, a negative bubble would imply that stock prices eventually become negative within a finite time horizon. Hence, negative bubbles are inconsistent with rational expectations and an infinite investment horizon but we cannot rule out (rational) bubbles when speculative investors focus on a limited time horizon. The simplest example of a process that satisfies (4) is the deterministic bubble, given by $B_t = (1 + R)^t B_0$, where B_0 is an initial value. A somewhat more realistic example, in which the bubble does not necessarily grow forever was suggested by Blanchard and Watson (1982). Their bubble process is given by

$$B_{t+1} = \begin{cases} \pi^{-1}(1 + R)B_t + \mu_{t+1}, & \text{with probability } \pi \\ \mu_{t+1}, & \text{with probability } 1 - \pi \end{cases} \tag{5}$$

where $\{\mu_t\}_{t=1}^{\infty}$ is an independent and identically distributed (iid) sequence of random variables with zero mean. In each period, the bubble generated as in equation (5) will continue with probability π, or collapse with probability $1 - \pi$. As long as the bubble does not collapse, the realized bubble return exceeds the interest rate R as a compensation for the risk of financial losses during a collapse

of the bubble.

Time Series Properties of Stock Prices and Dividends

Starting with Bachelier (1900) it is typically assumed that stock prices can be well approximated by a martingale process of the form

$$E_t(P^f_{t+1}) \;=\; P^f_t.$$

There are two possible rationales to motivate this assumption. First, assuming that D_t is a random walk with drift

$$D_{t+1} = \mu + D_t + \varepsilon_{t+1}$$

it follows that

$$P^f_t \;=\; \frac{(1+R)\mu}{R^2} + \frac{1}{R}D_t \;.$$

Thus, the fundamental value of the stock is also characterized by a random walk with drift. If $\mu \approx 0$, then $E_t(P^f_{t+1} - P^f_t) \approx E(\epsilon_{t+1})/R = 0$. Although empirical studies typically find that dividends are well approximated by a random walk, there is no convincing reason why this should always be the case.

In many applications the sampling frequency of stock prices is higher than the dividend period. Assume for example that dividends are payed out annually, whereas stock prices form a monthly series. Accordingly, the dividend series D_t is zero in eleven out of 12 months. During these 11 months the "perfect forecast present value" of future dividends $P^*_{t+1} \;=\; \sum_{i=1}^{\infty} \left(\frac{1}{1+R} \right)^i D_{t+i}$ does not change during periods without dividend payment (i.e. $P^*_{t+1} = P^*_t$). Therefore, the change in stock price during periods without dividend payment is solely due

to the update of expectations about the future stream of dividends, that is,

$$P_t^f - P_{t-1}^f = E_t(P_{t+1}^*) - E_{t-1}(P_{t+1}^*).$$

If it is assumed that investors' expectations are rational, it follows that updates of expectations from period t to $t+1$ form a martingale difference sequence with

$$E_t(P_{t+1}^f - P_t^f) = 0$$

and, thus, P_t^f can be represented by a random walk. In time periods with dividend payment the change of the fundamental value results as

$$P_t^f - P_{t-1}^f = E_t(P_{t+1}^*) - E_{t-1}(P_{t+1}^*) + D_t .$$

Accordingly, the dividend adjusted price series $P_t - D_t$ (where D_t is zero for periods without dividend payment) can be represented by a random walk.[1]

Diba and Grossman (1988) and Campbell and Shiller (1987) argue that in the absence of bubbles, prices and dividends are cointegrated, that is, prices and dividends are driven by a common stochastic trend. This can be seen by noting that

$$S_t \equiv P_t^f - \frac{1}{R}D_t = E_t(S_{t+1}^*)$$

$$\text{with} \quad S_{t+1}^* = \sum_{i=1}^{\infty} \left(\frac{1}{1+R}\right)^i \Delta D_{t+i}.$$

If ΔD_t is stationary and $R > 0$, then S_t^* is stationary as well. Since the (rational expectation) forecast of a stationary variable must also be stationary, it follows that under the assumptions of the present value model for stock prices $S_t = P_t^f -$

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[1] Note that on exchange markets the stock price immediately adjusts for dividends so that the $P_t - D_t$ represents the *ex dividend* price of the share.

$R^{-1}D_t$ is a stationary linear combination of stock prices and dividends. As shown by Campbell and Shiller (1987) the model implies that also ΔP_t^f is a stationary process and, thus, the vector (P_t^f, D_t) is cointegrated in the terminology of Engle and Granger (1987).

Diba and Grossman (1988) apply unit root tests to prices and dividends from the S&P Composite Stock Price Index from 1871 to 1986 and find that prices and dividends are both $I(1)$. Furthermore, cointegration tests suggest that stock prices and dividends are indeed cointegrated implying no evidence for an explosive bubble component in stock prices.

Backward-Looking Tests for a Structural Break

A natural empirical approach to identify speculative bubbles is to apply statistical tests for a structural change from a random walk to an explosive regime. Such tests were originally proposed by Phillips, Wu, and Yu (2011) (henceforth: PWY) and further developed by Phillips and Yu (2011), Homm and Breitung (2012), and Phillips, Shi, and Yu (2013).

The simplest version of the test is based on a first order autoregressive process

$$y_t = \rho y_{t-1} + \epsilon_t \ , \tag{6}$$

where $E(\epsilon_t) = 0$ and $E(\epsilon_t^2)$ for $t = 1, \ldots, T$. The null hypothesis is that y_t follows a simple random walk, i.e.

$$H_0: \quad \rho = 1. \tag{7}$$

Under the alternative hypothesis y_t starts as a random walk but changes at time

$T^* = \tau^* T$ into an explosive regime:

$$H_1: \quad y_t = \begin{cases} y_{t-1} + \epsilon_t & , \text{ for } t = 1, \ldots, \tau^* T \\ \rho y_{t-1} + \epsilon_t & , \text{ for } t = \tau^* T + 1, \ldots, T \quad \text{with } \rho > 1 \end{cases} \quad . \tag{8}$$

Assume for the moment that the true break fraction (or relative break date) $\tau^* = T^*/T$ is known. Then it is straightforward to test the hypothesis of a structural break at period T by a Chow test for the parameter $\phi = \rho - 1$ which is zero before T^* and positive after T^*. This gives rise to the regression function

$$\Delta y_t = \delta\, y_{t-1} D_t(\tau^*) + \varepsilon_t, \tag{9}$$

where $D_t(\tau^*)$ is a dummy variable which is zero for $t < \tau^* T$ and changes to one for $t \geq \tau^* T$. Correspondingly, the null hypothesis of interest $H_0 : \delta = 0$ is tested against the (one-sided) alternative $H_1 : \delta > 0$ by using an ordinary t-statistic:

$$\text{DFC}_{\tau^*} = \frac{\sum\limits_{t=\tau^* T+1}^{T} \Delta y_t y_{t-1}}{\widetilde{\sigma}_{\tau^*} \sqrt{\sum\limits_{t=\tau^* T1}^{T} y_{t-1}^2}} \;,$$

where

$$\widetilde{\sigma}_{\tau^*}^2 = \frac{1}{T-2} \sum_{t=2}^{T} \left(\Delta y_t - \widehat{\delta}_{\tau^*} y_{t-1} D_t(\tau^*) \right)^2$$

and $\widehat{\delta}_{\tau^*}$ denotes the OLS estimator of δ in (9). This test is labeled as Chow-type Dickey-Fuller (DFC) test in Homm and Breitung (2012).

An important problem is that in practice the absolute (T^*) or relative (τ^*) break date is usually unknown. Following Andrews (1993) a test with unknown break date can be constructed by trying out all possible break dates τ^* within a reasonable interval $\tau^* \in [\tau_0, 1 - \tau_0]$, where τ_0 is often set to 0.10 or 0.15. The

reason for leaving out break dates at the beginning and end of the example is that the regression (9) performs poorly if the number of observations in one of the two regimes is too small. The test statistic for an unknown break date is the maximum of all $T(1 - 2\tau_0)$ DFC statistics, labelled as supDFC. The limiting distribution of this test statistic can be expressed as a supremum of a function of Brownian motions (cf. Homm and Breitung (2012)).

It is important to note that under the alternative it is assumed that the bubble runs up to the end of the sample, that is, the price series does not switch back from the bubble regime into the random walk regime. As indicated by the some Monte Carlo simulations presented in Homm and Breitung (2012) the power of the test may deteriorate dramatically if the sample includes the collapse of the bubble. Phillips, Shi, and Yu (2013) suggest a search strategy for three and more regimes which is computationally demanding and may imply a severe loss in power. A simpler solution is to adapt the sample such that it focuses on the upswing of stock prices only. Obviously this strategy involves the risk of data mining which may bias the test result towards finding a bubble.

Real Time Monitoring

Classical test procedures like the sequential DFC statistic are designed to detect speculative bubbles within a given historical data set. Accordingly, these tests can be employed to answer the question whether a bubble occurs in a particular time span (e.g. the dot.com bubble in the late 1990s). From a practical point of view it is often more interesting to analyze whether some asset class is *currently* characterized by a speculative bubble. To this end a sequence of statistics is constructed that summarizes the accumulated evidence for a bubble at each point in time. If the statistic exceeds some threshold, we are able to conclude that with a prespecified probability the price series has entered an explosive regime. An

important advantage of this approach is that the test statistic is not affected by a later collapse of the bubble.

Assume that, when the monitoring starts, a training sample of n observations is available and that the null hypothesis of no structural break holds for the training sample. Then, in each period $n + 1, n + 2, \ldots$, a new observation arrives. Following Chu, Stinchcombe, and White (1996) we consider two different statistics (detectors):

$$\text{CUSUM:} \qquad S_r = \frac{1}{\widehat{\sigma}_r} \sum_{j=n+1}^{r} (y_j - y_{j-1}) = \frac{1}{\widehat{\sigma}_r} (y_r - y_n) \quad (r > n) \qquad (10)$$

$$\text{FLUC:} \qquad \text{DF}_r = \frac{1}{n+r} (\widehat{\rho}_r - 1) / \widehat{\sigma}_{\rho_r} \qquad (11)$$

where $\widehat{\sigma}_r^2$ is some consistent estimator of the residual variance, $\widehat{\rho}_r$ denotes the OLS estimate of the autoregressive coefficient (including a constant) based on the subsample $1, \ldots, n, n + 1, \ldots, r$ and $\widehat{\sigma}_{\rho_r}$ denotes the associated standard error. Assume that we enter the bubble regime at some period $T^* > n$. Then for $r > T^*$ the estimate $\widehat{\rho}_r$ tends to become large and eventually exceed the critical level c_α. Similarly, within the bubble regime the $r - n$ step ahead forecast error $y_r - y_n$ based on a random walk forecast tends to large positive values. PWY derived a fixed critical value for the FLUC detector, which implies that at a significance level of 0.05 a bubble is detected if $\text{DF}_r > 1.468$. For the CUSUM detector Chu, Stinchcombe, and White (1996) developed a time dependent critical value $c_\alpha(r) = \sqrt{b_\alpha + \log(r/n)}$, where for the one-sided bubble test Homm and Breitung (2012) obtained $b_{0.05} = 4.6$. In their Monte Carlo simulations, Homm and Breitung (2012) found that the FLUC detector tends to outperform the CUSUM detector.

Break Date Estimation

An important practical problem is to date stamp the bubble start. The sequential Dickey-Fuller test (which is equivalent to the FLUC monitoring approach) proposed by PWY can straightforwardly be used as an estimation procedure for the bubble start. The estimate for the starting date of the bubble is the smallest value r such that DF_r is greater than the right-hand critical value of the asymptotic distribution of the standard Dickey-Fuller t-statistic. In order to obtain a consistent estimator of the break date the critical value is specified as a function of the sample size T. Specifically, PWY proposed to apply time dependent critical values for estimating the emergence and crash of the bubble computed as $c_r = \log[\log(r)]/100$ for $r = n+1, \ldots, T$. Homm and Breitung (2012) pointed out that this approach involves a substantial delay, which can be avoided by using the ML estimator proposed by Bai (1994). The ML estimator results in minimizing the sum of squared residuals which is equivalent with the maximum of the DFC statistic. In order to date stamp the end of a bubble episode Breitung and Kruse (2013) consider tests for a change from a bubble regime into a random walk regime. Accordingly such test procedures reverse the null hypothesis and alternative of the Chow test considered in Section . To this end we run the usual Chow for the null hypothesis $H_0 : \rho_t \geq 1$ versus the alternative $H_1 : \rho_t = 1$ by running the regression

$$y_t = \varrho y_{t-1} + \phi y_{t-1} D_t(\tau) + e_t \quad t = 2, 3, \ldots, T, \tag{12}$$

where $D_t(\tau)$ is defined as in Section (). Unfortunately, under the null hypothesis of an explosive process the t-statistic for the hypothesis $\phi = 0$ possesses a degenerate limiting distribution as it converges to zero in probability. To sidestep this difficulty Breitung and Kruse (2013) develop a simple modification that yields

a test statistic with a standard normal limiting distribution. This modified test statistic allows us to test whether the autoregressive coefficient switches to a lower value within some range of possible break dates in the interval $\tau \in [\tau_0, 1 - \tau_0]$. Again, the break date τ^* that minimizes the residual sum of squares in the (12) is the ML estimator for the end date of the bubble. This estimator avoids the build-in time lag of the date stamping procedure proposed by PWY and Phillips, Shi, and Yu (2013).

An Empirical Example

In this section I illustrate the econometric methods for analyzing the well known "dot.com bubble" in the late 1990s.[2] A visual inspection of the (real) monthly NASDAQ Composite price index and dividends already suggest that the accelerating upswing starting with 1995 cannot be explained by fundamental factors alone (see Figure 1). Indeed, as shown in Figure 2, the sequential Dickey-Fuller statistic (the FLUC detector for the monitoring procedure) rejects the null hypothesis of no speculative bubble in the NASDAQ index from June 1995 up to May 2001. Thus according to PWY the bubble runs for years. Applying the sequential Chow statistic considered in Section we obtain the test statistic $supDFC=$ 3.0467, which is significant according to the 0.05 critical value of 1.296. In contrast, the $supDFC$ statistic is insignificant for the dividend series. Furthermore, the maximum of the DFC statistic is obtained for February 1995, four months before the sequence of Dickey-Fuller statistic becomes significant. This demonstrates the delay of the latter approach when date stamping the bubble.

Applying the Chow test for a collapse of the bubble we are able to reject the hypothesis that the bubble continuous at March 2000, more than one year before the sequential Chow statistic becomes insignificant. Note that during these 14

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[2]The empirical results presented in this section are taken from Homm and Breitung (2012) and Breitung and Kruse (2013).

months the price index looses 50 percent of its value! This demonstrates that for a practical implementation, a timely signal for the collapse of the bubble is essential. An investor who was perfectly riding the bubble in the NASDAC Composite index receives an annual return of 28 percent per year (on top of the dividends).

Conclusion

In recent years various econometric tools for analyzing speculative bubbles were developed. Among these methods, tests for a switch from a fundamental to a bubble regime (and vice verse) are particularly promising. These tests are easy to apply and have been shown to provide reliable and powerful evidence for or against a bubble. An obvious drawback of these tests is that they do not incorporate information on the fundamental value of the price series but merely summarize the evidence for an explosive path of the price series. By analyzing the difference between the actual and fundamental prices we should be able to design more powerful tests for persistent and accelerating mispricing of the asset which is due to a speculative excess.

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Appendix

Figures
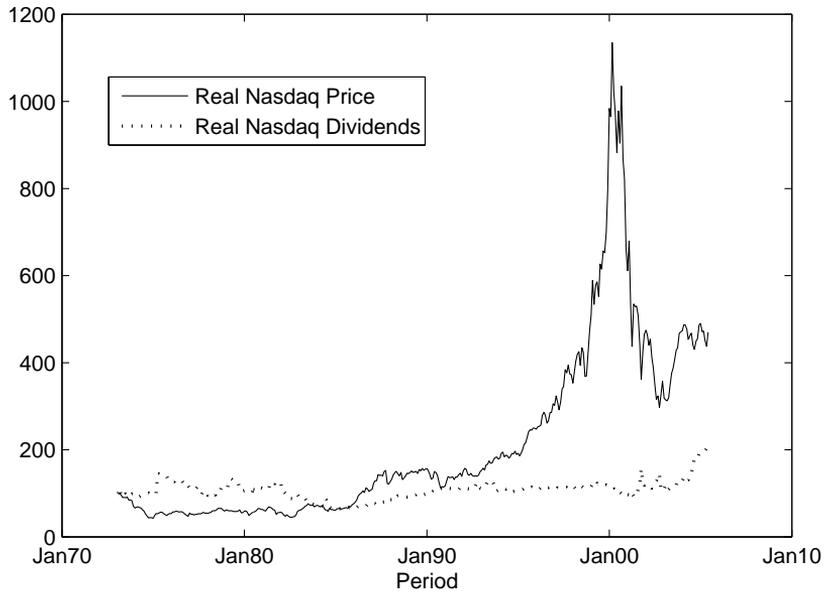
Note: Both series were normalized to 100 at the beginning of the period.
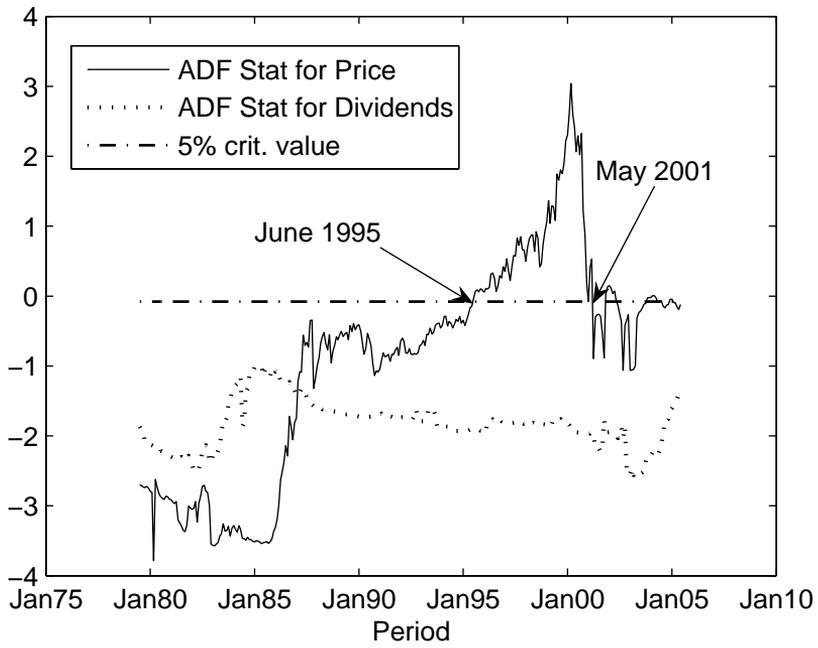
Figure 1: Real Nasdaq Price and Dividends (normalized)

Figure 2: DF$_r$ statistic for real Nasdaq price index and dividends