An Introduction to Barrier Options — Closed Form Solution and a Monte Carlo Approach

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Introduction

In recent years barrier options have become increasingly popular and frequently traded financial instruments, especially appearing in retail-products, so-called "certificates", broadly offered in the German retail market. Barrier options are path-dependent exotic options that become activated or null if the underlying reaches certain levels. There are four main types of barrier options that can either have call or put feature: Down-and-In, Down-and-Out, Up-and-In and Up-and-Out. The "down" and "up" refer to the position of the barrier relative to the initial underlying price. The "in" and "out" specify the type of the barrier, referring to activating and nullifying when the barrier is breached respectively. Barrier options always come at a cheaper price than ordinary options with same features (Taleb and Proß-Gill (1997)). A Down-and-Out call option for instance becomes nullified if the price of the underlying falls below the barrier. Despite being frequently traded nowadays, barrier options are still known as exotic options since they cannot be replicated by a finite combination of standard products,

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i.e. vanilla call and put options, future contracts etc. (Hausmann, Diener, and Käsler (2002)). Already in 1973, Robert C. Merton described in his article (Merton (Spring, 1973)) a closed form solution for the price of a Down-and-Out call option. Since then the market for barrier options literally exploded. This paper gives an introduction to barrier options and its properties and derives the analytic closed form solution by risk-neutral valuation. Furthermore we apply Monte Carlo simulation to derive numerical results. The great advantage of Monte Carlo simulation lies in the fact that it is robust and can be easily extended to options depending on multiple assets when no analytical solutions exist (Moon (2008)). Simple Monte Carlo simulation faces the problem that it yields both high statistical and discretization errors due to the knockout feature of the barrier option. Thus we subsequently introduce two error reduction techniques, namely *Control Variates* and *Brownian bridges* to counter these problems. Throughout the paper we assume a filtered probability space $(\Omega, \mathcal{M}, \{\mathcal{F}_t\}_{t>0}, P)$ with respect to the filtration $\{\mathcal{F}_t\}_{t>0}$ where $W = W_t$ is a standard Brownian motion. Furthermore we assume a world satisfying the Black Scholes conditions where the money market account is described by $dB_t = rB_t dt$ and the underlying S follows a geometric Brownian motion model, i.e. $dS_t = \alpha S_t dt + \sigma S_t dW_t^P$ with W_t^P denoting a standard Brownian motion under the measure P and σ and α fixed. Our final goal guiding and motivating us through the whole paper is to price a very popular German retail product: The European bonus certificate. The payoff of such a certificate with strike price K and lower boundary H is:

$$\text{Payoff} = \begin{cases} S_T, & \text{if } \exists\, t: \ S_t \leq H \text{ or } S_T > K \\ K, & \text{else} \end{cases}$$

The payoff of such an ordinary bonus certificate is shown in Figure 1. Investors see bonus certificates as alternatives to direct investments into the underlying.

They offer the holder the chance to earn more than holding the underlying as long as the underlying stays between a strike price K and a boundary H with $K \geq H$. Those products are primarily purchased by investors who believe that the underlying will not fluctuate a lot. If this expectation turns out to be true the bonus certificate will yield a higher payoff than the underlying. A bonus certificate is a portfolio consisting of a zero-strike call and a long position in a Down-and-Out put option (Reinmuth (2002)): $p_{bc} = p_{dkop} + p_{zero\ call}$ where p_{bc} is the price of a bonus certificate, p_{dkop} that of a Down-and-Out put option and $p_{zero\ call}$ that of a zero-strike call option. Thus we regard a simple European Down-and-Out put option in the following. Nonetheless we keep in mind that there also exist "non-simple" barrier options such as multi-barrier options or barrier options that require the asset price to not only cross a barrier, but spend a certain length of time across the barrier in order to knock in or knock out. The analytical challenge in our case is to calculate p_{dkop}. Therefore we apply the technique of risk-neutral valuation and make use of deep results from stochastic calculus such as the *Reflection Principle* and *Girsanov's theorem* to calculate the expectation of the payoff under the risk-neutral measure and discount it with the risk-free spot rate similar to pricing a vanilla put option (Steele (2001)):

$$p_{dkop} = e^{-rT} E^Q \left[(K - S_T) \mathbf{1}_{\{K \geq S_T; \inf_{t \in [0;T]} S_t \geq H\}} \right]$$

with time to maturity T, strike K, barrier $H \leq K$ and Q a risk-neutral measure with the money market account as numéraire. In our numerical simulation part we simulate our asset price according to geometric Brownian motion and implement the barrier as nullifying condition for each path. Furthermore we introduce a variance reduction technique called Control Variates and a discretization error reducing technique exploiting the idea of Brownian bridges. We conclude with a discussion on the scope and limitations of the introduced techniques. The specific bonus certificate that we price in this paper is a Goldman Sachs certificate with ISIN DE000GS3DWL0 Goldman Sachs (2011a) and properties shown in Table 1 on May 8, 2011. The ask price of this European bonus certificate is **84.06** at day of pricing, so this price acts as our benchmark after taking into account limits of our model like the issuer risk and profit margins of Goldman Sachs.

Analytical Solution

Given linearity of the expectation the price of our option can be written as:

$$\begin{aligned} p_{dkop} &= e^{-rT}E^Q\left[(K-S_T)\mathbf{1}_{\{K\geq S_T;\inf_{t\in[0;T]}S_t\geq H\}}\right] \\ &= e^{-rT}\Big(E^Q\left[(K-S_T)\mathbf{1}_{\{S_T<K\}}\right]-E^Q\left[(K-S_T)\mathbf{1}_{\{S_T<H\}}\right] \\ &\quad -E^Q\left[(K-S_T)\mathbf{1}_{\{S_T>H;\inf_{t\in[0;T]}S_t<H\}}\right] \\ &\quad +E^Q\left[(K-S_T)\mathbf{1}_{\{S_T>K;\inf_{t\in[0;T]}S_t<H\}}\right]\Big) \end{aligned}$$

We immediately see that the first expectation is just the price of a plain-vanilla European put option with strike K that we know from Black (1973), yielding $e^{-rT}E^Q\left[(K-S_T)\mathbf{1}_{\{S_T<K\}}\right]=Ke^{-rT}N(-d_2)-S_0N(-d_1)$, with $d_1=\sigma\sqrt{T}+\frac{ln(\frac{S_0}{K})+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2=d_1-\sigma\sqrt{T}$. Applying the same risk-neutral valuation technique to the second expectation it takes us little effort to see that we only need to replace K by H in d_1 and d_2 to get the second expectation. Now we turn to the third and fourth term. Exemplarily we calculate the fourth expectation setting $m:=\frac{r-\frac{1}{2}\sigma^2}{\sigma}$, $h:=\frac{1}{\sigma}\ln\left(\frac{H}{S_0}\right)$ and $k:=\frac{1}{\sigma}\ln\left(\frac{K}{S_0}\right)$ for sake of readability. We plug in $S_t=S_0\exp\{(r-\frac{1}{2}\sigma^2)t+\sigma W_t^Q\}$ with $W_t^Q=W_t^P+\frac{\alpha-r}{\sigma}t$ being a Brownian motion under the measure Q (Hull (2007)) for S_T and S_t to

get:

$$
\begin{aligned}
& e^{-rT} E^{Q}\left[(K - S_T)\mathbf{1}_{\{S_T > K; \inf_{t\in[0;T]} S_t < H\}}\right] \\
&= e^{-rT}\Bigg(KE^{Q}\left[\exp\left\{mW_T^Q - \frac{1}{2}m^2T\right\}\mathbf{1}_{\{W_T^Q > k; \inf_{t\in[0;T]} W_t^Q < h\}}\right] \\
&- E^{Q}\left[\exp\left\{mW_T^Q - \frac{1}{2}m^2T\right\}S_0\exp\left\{\sigma W_T^Q\right\}\mathbf{1}_{\{W_T^Q > k; \inf_{t\in[0;T]} W_t^Q < h\}}\right]\Bigg)
\end{aligned}
$$

Now we calculate the joint distribution $Q(W_T^Q > k; \inf_{t\in[0,T]} W_t^Q < h)$ dismissing the *inf term* making use of the reflection principle. Plugging back m, h and k we get the desired expression:

$$
\begin{aligned}
&= e^{-rT}K\left(\frac{H}{S_0}\right)^{\frac{2r-\sigma^2}{\sigma^2}} N\left(\frac{\ln\left(\frac{H^2}{S_0K}\right) + rT\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right) \\
&- S_0\left(\frac{H}{S_0}\right)^{\frac{2r+\sigma^2}{\sigma^2}} N\left(\frac{\ln\left(\frac{H^2}{S_0K}\right) + rT + \frac{1}{2}\sigma^2T}{\sigma^2\sqrt{T}}\right)
\end{aligned}
$$

Finally we can put all four expectations together and get the price:

$$
\begin{aligned}
\mathbf{p_{dkop}} &= Ke^{-rT}N(-d_2) - S_0N(-d_1) + S_0N(-x_1) \\
&- Ke^{-rT}N(-x_1 + \sigma\sqrt{T}) - S_0\left(\frac{H}{S_0}\right)^{2\lambda}[N(y) - N(y_1)] \\
&+ Ke^{-rT}\left(\frac{H}{S_0}\right)^{2\lambda-2}[N(y - \sigma\sqrt{T}) - N(y_1 - \sigma\sqrt{T})], \qquad (1)
\end{aligned}
$$

with $\lambda := \frac{r+\frac{\sigma^2}{2}}{\sigma^2}$, $y := \frac{\ln\left(\frac{H^2}{SK}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$, $x_1 := \frac{\ln\left(\frac{S}{H}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$, $y_1 := \frac{\ln\left(\frac{H}{S}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$. Now let us turn back to our bonus certificate. By letting K going to zero we calculate the price of our zero-call on the DAX using the Black-Scholes formula for European call options (Hull (2007)) as 74.9225. Plugging the data of our bonus certificate into the above derived formula (1) for pricing European Down-and-Out put options we get: $\mathbf{p_{dkop}} = 9.4625$. In summing up the two prices we get the final price of the European bonus certificate as **84.39** which is close to the quoted price of **84.06**.

Numerical Simulation

For our Monte Carlo simulation we generate M underlying asset price paths $\{(S^1_{t_1}, ..., S^1_{t_n}), ..., (S^M_{t_1}, ..., S^M_{t_n})\}$ on a fixed set of points in time $0 < t_1 < ... < t_n = T$ for $k = 1, ..., M$ each following a geometric Brownian motion. We then determine the payoff of the barrier option conditional on the fact that no barrier-breach has occurred at any time step. Subsequently we take the discounted average of the payoffs to obtain the numerical price of our barrier option. This estimator is unbiased and converges with probability 1 as $n \to \infty$ and the statistical error is of order $\mathcal{O}(1/\sqrt{M})$. For Barrier Options with continuous knock-out observation the discretization error−here the hitting time error (i.e. missing a barrier-breach that happens between two simulated time steps)−is of order $\mathcal{O}(1/\sqrt{n})$ (Gobet (2000)). The algorithm is presented in Algorithm 1. In our case we choose a discretization of N = 1,000 equidistant time steps per year and simulate M = 20,000 paths in total for our Monte Carlo simulation. The resulting price is **9.4976** compared to the theoretical price of **9.4625**. The difference in value can firstly be explained by the fact that we have a very high variance of **89.06** (9.44 standard deviation) due to the knockout feature of the option. Secondly we have a discretization of only 1,000 time steps, so the option can only knockout on those nodes whereas our initial barrier option theoretically can knock out at any time. In the following we will tackle the first problem by reducing the variance of our results, the second problem will be addressed thereafter through the concept of Brownian bridges, where we estimate the probability of the option knocking out between two time steps.

Control Variates

After having introduced the simple Monte Carlo simulation we now turn our focus to improving the efficiency of our simulation. In the following we introduce a variance reducing technique called Control Variates (Glasserman (2003)). The idea behind this technique is to exploit information about the errors in estimates of known variables that have a dependence to our variable (in our case $\text{Payoff}^k := Y^k$) and thus reduce the variance of our variable. We therefore choose another output X_k (*the control variate*) that is correlated to our payoff and of which we know the expectation and regard $Y^k(b) = Y^k - b(X_k - E[X])$ for a fixed b. By minimizing $Var[Y^k(b)]$ we find that the optimal choice of b^* is given by $b^* = \frac{\sigma_Y}{\sigma_X}\rho_{XY} = \frac{Cov[X,Y]}{Var[X]}$, where ρ_{XY} stands for the correlation between X and Y. Subsequently we estimate $Cov[X, Y]$ and $Var[X]$ using least-squares regression to estimate b^* and arrive at our control variate estimator $\overline{Y}(b) = \overline{Y} - \hat{b}_M(\overline{X} - E[X]) = \frac{1}{M}\sum_{k=1}^{M}(Y^k - \hat{b}_M(X_k - E[X]))$. This estimator is consistent and unbiased if the correct b^* is known. The estimation of b^* however introduces some bias that vanishes quickly as M becomes sufficiently large (Glasserman (2003)). The algorithm is described in Algorithm 2. We start with a primitive control setting X_k as *iid* normally distributed random variables, then move forward to use the underlying asset as control variate and finally we examine a plain vanilla option as control variate. As our crude Monte Carlo paths are simulated from a sequence of independent standard normal random variables there exists some correlation between those random variables and our desired payoff. The correlations between underlying and payoff as well as between the plain vanilla option and payoff are straightforward. Implementing the primitive control in our initial model we find out that we get an option price of **9.4976** and the variance becomes **89.0632640717** (9.44 standard deviation) compared to **89.0632748101** as before. The variance reduction is minimal as we have very small correlation. But those generic control variates are always available in a simulation and are mostly easy to implement. In contrast, using the underlying price as control variate yields a variance of **17.02** (4.13 standard deviation) which is significantly better than the primitive control. The correlation and thus the effectiveness of the control variate depends on the strike and the barrier. When we regard our Down-and-Out put option we can imagine that the higher the strike is the greater $|\rho_{SY}|$ becomes. Also the lower our barrier is the more we expect the correlation to be high speaking in absolute terms. Furthermore we observe that with plain vanilla put options as control variate our model yields a variance of **2.9934e-24** (1.73e-12 standard deviation) and the resulting price does not differ from our theoretical value up to the **fifth decimal**. This kind of control is very effective if the barrier is low. In fact, if the barrier is set to be zero the correlation is very close to 1. Simulation results are summarized in Table 3.

Brownian Bridges

Now that we have introduced a powerful technique reducing the statistical error of our simulation we turn our focus to reducing the discretization error, in our case the hitting-time error. Hitting-time error refers to the error that arises from not sufficiently fine discretization, i.e. when a continuously observed barrier option knocks out between two simulated time steps, but the underlying asset subsequently recovers at the latter time step, formally: For simulated time steps t_i and t_{i+1} we have $S_{t_i} > H$ and $S_{t_{i+1}} > H$, but $S_t < H$ for at least one $t \in [t_i, t_{i+1}]$. If we cannot simulate a large number of time steps due to limited computational time or resources the hitting-time error can become substantially large. Inspired by Mannella (1999) Moon (2008) proposed in his paper an efficient technique using Brownian bridges and the uniform distribution to reduce this hitting-time error for an Up-and-Out call option. The idea is to calculate an exit probability for each pair of time steps, i.e. the probability that the underlying breaches a lower boundary between two time steps. We then use a uniformly distributed random variable to decide whether this probability is sufficiently large to let our option knock out. Formally we regard a domain $D = (H, \infty)$ and define the probability P_i that the process S exits D at $t \in [t_i, t_{i+1}]$ given that S_{t_i} and $S_{t_{i+1}}$ are in D. Then P_i is exactly the exit probability that we have described above. To calculate this exit probability we can use the law of Brownian bridges and get:

$$P_i = P\left[\min_{t \in [t_i, t_{i+1}]} S_t \leq H \middle| S_{t_i} = s_1, S_{t_{i+1}} = s_2\right] = \exp\left(-2\frac{(H - s_1)(H - s_2)}{\sigma^2(t_{i+1} - t_i)}\right)$$

with s_1 and s_2 in D. In our algorithm we sample for each time step t_i a uniformly distributed random variable $u_i \sim \mathcal{U}(0, 1)$. If in t_i the exit probability P_i exceeds u_i we dismiss the path and set the payoff as zero. The algorithm with Brownian bridges is presented in Algorithm 3. As the barrier option that we have discussed until now has a very low barrier, thus our simulation with control variates yields a very good result, we want to illustrate the Brownian bridge approach in an example where the barrier is tight. We regard a Down-and-Out barrier put option on the DAX with the properties in Table 2 again priced on May 8, 2011. Using our analytic formula (1) we get a theoretical price of **0.4313**. In this case the barrier is only about 7% below the actual price. As we can imagine the simulated price using simple Monte Carlo should be quite bad since it is very likely that the option undetectedly knocks out between two time steps. Again simulating with 20,000 paths and 1,000 time steps we see a price of **0.4936** for our barrier option which is more than 14% above the theoretical price. In contrast, applying the Brownian bridge technique introduced before we get a price of **0.4484** which represents a significant improvement against the simple

Monte Carlo simulation. Again summarized results are presented in Table 3.

Concluding Remarks

Using the example of a European bonus certificate we examined in this paper basic properties and the pricing of barrier options both analytically and numerically. We practically introduced two techniques to reduce variance of the simple Monte Carlo simulation on the one hand and to reduce discretization error on the other hand. We observe that our analytical price is different from the quoted price in the market. Several factors lead to the possible discrepancy. Adjusting for issuer risk, volatility skew, barrier shift and risk-free rate we expect to gain more accurate results. Also given our foundation it is not a difficult task to extend our model to pricing more complex barrier options, such as multi-barrier, Asian or even Parisian barrier options. For loose barriers we propose reducing variance by using appropriate control variates and going further various other numerical techniques like for instance importance sampling. For relatively tight barriers we propose reducing discretization error by using the Brownian bridge approach.

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Appendix

Algorithms

Algorithm 1. ***Standard Monte Carlo Method***

for $k = 1, ..., M$

for $i = 1, ..., n$

generate a $\mathcal{N}(0,1)$ *sample* Z_i

set $S_{t_{i+1}}^k = S_{t_i}^k \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\sqrt{t_{i+1} - t_i} + \sigma\sqrt{t_{i+1} - t_i}\ \ Z_i\right\}$

end

if $\max_{1 \le i \le n} S_{t_i} > H$ *then* $Payoff^k = \max(K - S_{t_n}^k)$

else $Payoff^k = 0$

end

set $p_{dkop} = \frac{1}{M}\sum_{k=1}^{M}\left(e^{-rT}Payoff^k\right)$

Algorithm 2. ***Control Variate Monte Carlo Method***

for $k = 1, ..., M$

generate X_k

for $i = 1, ..., n$

generate a $\mathcal{N}(0,1)$ *sample* Z_i

set $S_{t_{i+1}}^k = S_{t_i}^k \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\sqrt{t_{i+1} - t_i} + \sigma\sqrt{t_{i+1} - t_i}\ \ Z_i\right\}$

end

if $\max_{1 \le i \le n} S_{t_i} > H$ *then* $Y^k = \max\left(K - S_{t_n}^k\right)$

else $Y^k = 0$

end

set $\hat{b}_M = \frac{\sum_{k=1}^{M}(X_k - \overline{X})(Y^k - \overline{Y})}{\sum_{k=1}(X_k - \overline{X})^2}$

for $k = 1, ..., M$

  set $Y^k = Y^k - \hat{b}_M(X_k - E[X])$

end

set $p_{dkop} = \frac{1}{M}\sum_{k=1}^{M}\left(e^{-rT}Y^k\right)$

Algorithm 3. ***Brownian Bridges***

for $k = 1, ..., M$

  for $i = 1, ..., n$

    generate a $\mathcal{N}(0,1)$ *sample* Z_i

    set $S^k_{t_{i+1}} = S^k_{t_i}\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\sqrt{t_{i+1} - t_i} + \sigma\sqrt{t_{i+1} - t_i}\ \ Z_i\right\}$

    set $P_{t_{i+1}} = \exp\left\{-2\frac{(H - S_{t_i})(H - S_{t_{i+1}})}{\sigma^2(t_{i+1} - t_i)}\right\}$

  end

  generate a $\mathcal{U}(0,1)$ *sample* $u_i, i = 1, ..., n$

  if $S_{t_i} > H$ *and* $P_{t_i} < u_i, \forall i \ \ 1 \leq i \leq n$ *then* $Y^k = \max\left(K - S^k_{t_n}\right)$

  else $Y^k = 0$

end

set $p_{dkop} = \frac{1}{M}\sum_{k=1}^{M}\left(e^{-rT}Y^k\right)$

Figures

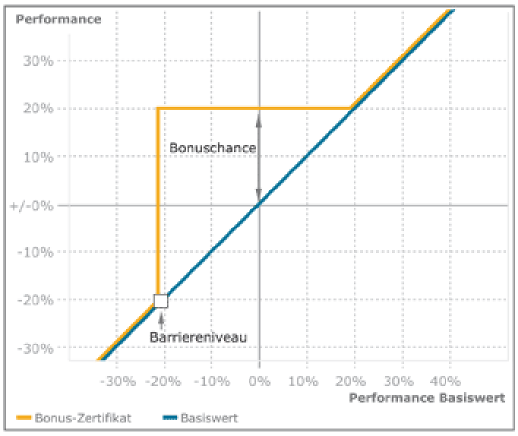

Figure 1: Payoff of a bonus certificate
Source: Goldman Sachs (2011c)

Tables

The underlying prices, the levels of the barriers and the exercise prices have been multiplied by 0.01.

Underlying	DAX Performance Index
Time to maturity	11 months
Exercise price (K)	82.5
Barrier (H)	27
Price of the underlying	74.9225
Credit Rating of the Issuer	A+/A1 (Fitch/Moody's) Goldman Sachs (2011d) CDS +71,08bp
Risk-free interest rate	1.38% (one year German government bond) Bloomberg (2011)
Implied volatility	18.2071% (annualized implied volatility of the DAX) Goldman Sachs (2011b)

Table 1: Bonus Certificate

Underlying	DAX Performance Index
Time to maturity	1 Year
Exercise price (K)	82.5
Barrier (H)	70
Price of the underlying	74.9225
Risk-free interest rate	1.38% (one year German government bond) Bloomberg (2011)
Implied volatility	18.2071% (annualized implied volatility of the DAX) Goldman Sachs (2011b)

Table 2: Second Barrier Option

First Barrier Option	**Option Price**	**Bonus Certificate**	**Variance**
Quoted	NA	84.06	-
Analytical	9.4625	84.39	-
Simple MC Simulation	9.4976	84.42	89.06
Primitive Control	9.4976	84.42	89.06
Underlying as Control	9.4906	84.41	17.02
Vanilla Option as Control	9.4625	84.39	2.99e-24
Second Barrier Option			
Analytical	0.4313	-	-
Simple MC Simulation	0.4808	-	3.02
Brownian Bridge Monte Carlo	0.4465	-	2.68

Table 3: Summary of Results